Inequality with extended angle bisectors.

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628. Proposed by Roland H. Eddy, Memorial University of Newfoundland.

Given a triangle ABC with sides $a,b,c,$ let T_a,T_b,T_c denote the angle bisectors extended to the circumcircle of the triangle. If R and r are the circum-and in-radii of the triangle, prove that

$$T_a+T_b+T_c\le 5R+2r,$$

with equality just when the triangle is equilateral.

Solution by Arkady Alt, San Jose ,California, USA.

Let AA_1 be angle bisectors extended to the circumcircle of the triangle.

Since $\angle AA_1C=B$ then $\angle A_1CA=180^\circ-\left(B+\frac{A}{2}\right)=90^\circ+\frac{B-C}{2}$ and, therefore,

by Sine Theorem $T_a=2R\sin\left(90^\circ+\frac{B-C}{2}\right)=2R\cos\frac{B-C}{2}$.

Thus, $T_a+T_b+T_c\le 5R+2r \Leftrightarrow \sum\cos\frac{B-C}{2}\le\frac{5}{2}+\frac{r}{R}\Leftrightarrow$

(**1**) $\quad\sum\cos\frac{B-C}{2}\le\frac{5}{2}+4\prod\sin\frac{A}{2}$.

Let $\alpha:=\frac{\pi-A}{2},\beta:=\frac{\pi-A}{2},\gamma:=\frac{\pi-A}{2}$ then $\alpha,\beta,\gamma\in(0,\pi/2),\alpha+\beta+\gamma=\pi.$

Let s,R and r be, respectively, semiperimeter, circumradius and inradius of some acute triangle with angles $\alpha,\beta,\gamma.$ Then inequality (**1**) becomes

$\sum\cos(\beta-\gamma)\le\frac{5}{2}+4\prod\cos\alpha\Leftrightarrow\sum\cos\alpha\cdot\cos\beta+\sum\sin\alpha\cdot\sin\beta\le\frac{5}{2}+4\prod\cos\alpha\Leftrightarrow$

$\frac{s^2+r^2-4R^2}{4R^2}+\frac{s^2+4Rr+r^2}{4R^2}\le\frac{5}{2}+4\cdot\frac{s^2-(2R+r)^2}{4R^2}.$

Since $s^2\ge 2R^2+8Rr+3r^2$ (Walker's Inequality) and $R\ge 2r$ (Euler's Inequality)

we obtain $\frac{5}{2}+4\cdot\frac{s^2-(2R+r)^2}{4R^2}-\frac{s^2+r^2-4R^2}{4R^2}-\frac{s^2+4Rr+r^2}{4R^2}=$

$\frac{s^2-R^2-10Rr-3r^2}{2R^2}\ge\frac{2R^2+8Rr+3r^2-R^2-10Rr-3r^2}{2R^2}=\frac{1}{2R}(R-2r)\ge 0$